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Irreducibility of the 3-D Stochastic Navier–Stokes Equation

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A 3-dimensional Navier–Stokes equation with random force is investigated. A form of irreducibility, of interest in ergodic theory, is proved, under a full noise assumption. The basic tool is the fact that, even if the equation is a priori non-well-posed, the solutions depend continuously on the noise around regular solutions. © 1997

1. INTRODUCTION

In this paper we consider a 3-dimensional Navier–Stokes equation with random body force. The aim is to prove that, if the noise affects all the different modes, then the probability distribution of any weak solution at any positive time is full in the energy space H; i.e., its support in the H-topology is H itself. This is a property of irreducibility, in the language of ergodic theory. We can prove this result only if the initial condition is in the Sobolev space H^1, but perhaps this technical condition can be avoided with more clever estimates.

This result relies on a controllability property and on the continuity of the mapping *noise* $\mapsto$ *solution* along the controllers. The most relevant fact seems to be the latter property, taking into account that it is not known if the 3-dimensional Navier–Stokes equation is well posed. It is well known that regular solutions are unique also in the class of weak solutions; similarly, the solution depends continuously on data, in the class of weak solutions, around regular solutions. We use this fact, along with the regularity of the trajectories involved in the controllability argument. In view of the previous remarks, it seems that the irreducibility result proved in this paper is typical of equations which are well posed, in a sense, around regular solutions, while it does not hold for any differential equation just as a consequence of the assumption that the noise affects all modes.

The present paper may be related conceptually to some investigations of Fursikov (see for instance [19, 20]), although we cannot do any more precise comparison.

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The main reason for the study of the irreducibility property is its relevance in ergodic theory, and more precisely in the analysis of the uniqueness and ergodicity of invariant measures. See, for instance, the classical works [10, 22], and the recent developments of these ideas for stochastic infinite dimensional systems [9, 26, 29, 31, 32]. In particular, it has been proved that the 2-D stochastic Navier–Stokes equation is ergodic [11, 15], a fact that is at the foundation of statistical fluid mechanics [27], and also, for instance, of the numerical methods based on empirical orthogonal functions [21]. The property of irreducibility in itself, is not sufficient to obtain any ergodic result, but it seems to be at the core of the problem; additional properties, like the strong Feller property, are needed, and we hope that results in this direction will be proved also for the 3-D stochastic Navier–Stokes equation, in the future.

1.1. *The Navier–Stokes Equation*

We consider a viscous incompressible homogeneous Newtonian fluid in a bounded open domain $D \subset \mathbf{R}^3$ with smooth boundary ∂D, described by the classical Navier–Stokes equations

$$
\begin{aligned}
&\frac{\partial u}{\partial t} + (u \cdot \nabla)\, u + \nabla p = \nu\, \Delta u + f + \frac{\partial \omega}{\partial t} && \text{in} \quad [0, T] \times D \\
&\operatorname{div} u = 0 && \text{in} \quad [0, T] \times D \\
&u = 0 && \text{in} \quad [0, T] \times \partial D \\
&u(0, x) = u_0(x), && x \in D
\end{aligned}
$$

(u is the velocity field, p the pressure field, $\nu > 0$ the kinematic viscosity, $f + \partial\omega/\partial t$ the body force). The term $\partial\omega/\partial t$ may be a white noise. A possible interpretation of the force $f + \partial\omega/\partial t$ is that it is composed of an "average" term f and a rapidly fluctuating part $\partial\omega/\partial t$.

We rewrite the previous equation in the usual abstract form. We set

$$
H = \{\phi : D \to \mathbf{R}^3 : \phi \in [L^2(D)]^3, \operatorname{div} \phi = 0, \phi \cdot n|_{\partial D} = 0\},
$$

where n is the outer normal to ∂D (cf. [34] for more details, and in particular for the interpretation of the condition $\phi \cdot n|_{\partial D} = 0$), and we set

$$
V = \{\phi \in [H^1(D)]^3 : \operatorname{div} \phi = 0, \phi|_{\partial D} = 0\}
$$

($H^\alpha(D)$ denotes the classical Sobolev space, see [25]). We denote by $|\cdot|$ and $\langle \cdot, \cdot \rangle$ the norm and inner product in H. Identifying H with its dual space H', and identifying H' with a subspace of V' (the dual space of V), we have $V \subset H \subset V'$ and we can denote the dual pairing between V and V'

by $\langle \cdot, \cdot \rangle$ when no confusion may arise. Moreover, we set $D(A) = [H^2(D)]^3 \cap V$, we denote by $D(A^{-1})$ the dual space of $D(A)$, and we perform identifications as above to get the dense continuous inclusions

$$D(A) \subset V \subset H \subset V' \subset D(A^{-1}).$$

With $D(A)$ defined above, we define the linear operator $A: D(A) \subset H \to H$ as $Au = -P\Delta u$, where P is the orthogonal projection in $[L^2(D)]^3$ over H. The operator A is positive selfadjoint with compact resolvent (see [36, Chap. III, Section 2.1]); we denote by $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$ the eigenvalues of A, and by $e_1, e_2, ...$ a corresponding complete orthonormal system of eigenvectors.

The fractional powers A^α of A, $\alpha \geqslant 0$, are simply defined by

$$A^\alpha x = \sum_{i=1}^{\infty} \lambda_i^\alpha \langle x, e_i \rangle \, e_i$$

with domain

$$D(A^\alpha) = \{x \in H: \|x\|_{D(A^\alpha)} < \infty\},$$

where

$$\|x\|^2_{D(A^\alpha)} = \sum_{i=1}^{\infty} \lambda_i^{2\alpha} \langle x, e_i \rangle^2 = |A^\alpha x|^2.$$

The space $D(A^\alpha)$ is an Hilbert space witch the inner product $\langle x, y \rangle_{D(A^\alpha)} = \langle A^\alpha x, A^\alpha y \rangle$, $x, y \in D(A^\alpha)$.

Since V coincides with $D(A^{1/2})$ (see [35, Section 2.2], or [36, Chap. III, Section 2.1]), we can endow V with the norm $\|u\| = |A^{1/2} u|$.

We remark that

$$\|u\|^2 \geqslant \lambda_1 \, |x|^2. \tag{1}$$

We define the bilinear operator $B(u, v): V \times V \to V'$ as

$$\langle B(u, v), z \rangle = \int_D z(x) \cdot (u(x) \cdot \nabla) \, v(x) \, dx$$

for all $z \in V$ (the integral is well defined since $V \subset [L^4(D)]^3$ by Sobolev embedding Theorem; see also (3) below). This operator can be extended in different topologies. By the incompressibility condition we have

$$\langle B(u, v), v \rangle = 0, \qquad \langle B(u, v), z \rangle = -\langle B(u, z), v \rangle. \tag{2}$$

We introduce two constants, C_B, C_S, which will play a basic role. By Hölder inequality, there exists a constant $C_B > 0$ such that

$$|\langle B(u, v), z\rangle| \leqslant C_B \|v\| \, |u|_{L^4(D)} \, |z|_{L^4(D)} \tag{3}$$

for all $v \in V$ and $u, z \in [L^4(D)]^3$, where $|\cdot|_{L^4(D)}$ denotes the classical norm in $[L^4(D)]^3$. Moreover, we have $H \subset [L^2(D)]^3$, $D(A) \subset [H^2(D)]^3$ (continuous injections), then by interpolation $D(A^{3/8}) \subset [H^{3/4}(D)]^3$. By Sobolev embedding Theorem, $[H^{3/4}(D)]^3 \subset [L^4(D)]^3$. Hence, $D(A^{3/8}) \subset [L^4(D)]^3$. Therefore, by a classical interpolation inequality, there exists a constant $C_S > 0$ such that

$$|u|_{L^4(D)} \leqslant C_S \|u\|^{3/4} \, |u|^{1/4} \tag{4}$$

for all $u \in V$ (cf. [35], Section 2.2).

With these preliminaies, we consider the following abstract version of the stochastic Navier–Stokes equation:

$$\begin{cases} \dfrac{du(t)}{dt} + Au(t) + B(u(t), u(t)) = f + \dfrac{d\omega(t)}{dt}, & t \in [0, T] \\ u(0) = u_0. \end{cases} \tag{5}$$

To avoid a useless change of assumption later, we assume throughout the paper that

$$f \in H.$$

1.2. *Definition of Solution*

Notice that in this subsection we shall introduce three sets Ω, Ω_0, Ω_{00} used throughout the paper.

We denote by Ω the space of all continuous functions ω: $[0, T] \to H$ that vanish at $t = 0$, endowed with the usual uniform topology, and we denote by $\mathscr{F}$ the corresponding Borel σ-algebra.

Let us take two numbers $s \in (0, \frac{1}{2})$ and $p \in (1, \infty)$ such that $s - (1/p) > \frac{3}{8}$ ($\frac{3}{8}$ is the Sobolev exponent that appears in the embeddings of the previous subsection; the upper bound $s < \frac{1}{2}$ could be avoided, but this would be misleading in view of the main application to Wiener measure). Let $W^{s,p}(0, T; H)$ be the space (cf. [1]) of all measurable functions f: $[0, T] \to H$ such that

$$\|f\|^p_{W^{s,p}(0,T;H)} := \int_0^T |f(t)|^p \, dt + \int_0^T \int_0^T \frac{|f(t) - f(r)|^p}{|t - r|^{1+sp}} \, dt \, dr < \infty.$$

By Sobolev Embedding Theorem, $W^{s,p}(0, T; H) \subset C^\alpha([0, T]; H)$, with continuous injection, for some $\alpha \in (\frac{3}{8}, \frac{1}{2})$($C^\alpha([0, T]; H)$ denotes the space of all α-Hölder continuous functions from $[0, T]$ to H). We denote by Ω_0

the space of all functions $\omega \in W^{s,p}(0, T; H)$ that vanish at $t=0$. We have $\Omega_0 \subset \Omega$, and we also have $\Omega_0 \in \mathscr{F}$ (this can be checked using the measurability, for every positive integer n, of the real valued mapping

$$\omega \to \int_0^T \int_0^T \frac{|\omega(t) - \omega(r)|^p}{(1/n) + |t-r|^{1+sp}} \, dt \, dr$$

defined on $(\Omega, \mathscr{F})$).

Let P be a measure on $(\Omega, \mathscr{F})$ such that

$$P(\Omega_0) = 1.$$

EXAMPLE 1.1. The main example of P is a Wiener measure on $(\Omega, \mathscr{F})$ (cf. [9]) for details, and in particular the proof of Theorem 3.3). Without loss of generality we can assume that P is the measure on $(\Omega, \mathscr{F})$ induced by the process $W(t, q) = \sum_{n=1}^{\infty} \sigma_n \beta_n(t, q) \phi_n$, $t \geqslant 0$, $q \in \mathscr{Q}$, where $\{\beta_n(t, q)\}$ is a sequence of independent one dimensional Bownian motions defined on a probability space $(\mathscr{Q}, \mathscr{G}, \mathscr{P})$, $\{\phi_n\}$ is a complete orthonormal system in H, and $\sum_{n=1}^{\infty} \sigma_n^2 < \infty$. We shall refer later to such a representation.

DEFINITION 1.1. We say that a measurable mapping $\omega \mapsto u(\cdot, \omega)$ from Ω to $L^2(0, T; H)$ is a generalized solution of Eq. (5) if there exists a measurable set $\Omega_{00} \subset \Omega_0$ with $P(\Omega_{00}) = 1$ such that for all $\omega \in \Omega_{00}$

$$u(\cdot, \omega) \in L^2(0, T; [L^4(D)]^3) \cap L^\infty(0, T; H) \cap C([0, T]; D(A^{-1}))$$

and satisfies the identity

$$\langle u(t), \phi \rangle + \int_0^t \langle u(s), A\phi \rangle \, ds - \int_0^t \langle B(u(s), \phi), u(s) \rangle \, ds$$

$$= \langle u_0, \phi \rangle + \int_0^t \langle f, \phi \rangle \, ds + \langle \omega(t), \phi \rangle \tag{6}$$

for all $\phi \in D(A)$ and all $t \in [0, T]$.

Note that the condition $\Omega_{00} \subset \Omega_0$ is not restrictive, since $P(\Omega_0) = 1$. We remark that the property $u(\cdot) \in L^\infty(0, T; H) \cap C([0, T]; D(A^{-1}))$ implies that $u(\cdot) \in C([0, T]; H_w)$ (H_w is the space H endowed with the weak topology). The proof can be found in [30]. Notice also that $\langle B(u(s), \phi), u(s) \rangle$ is well defined and integrable, by (3) and the assumption $u(\cdot, \omega) \in L^2(0, T; [L^4(D)]^3)$. Moreover, by (2), this term is formally equal to $-\langle B(u(s), u(s)), \phi \rangle$, so that (6) formally corresponds to an integrated version of (5).

The existence of solutions in that sense has been proved by many authors, under different assumptions, and with additional properties of the solutions (for the existence of solutions and other properties see, for instance, [2–8, 12, 14, 16–18, 22, 37, 38]). As in the deterministic case, the uniqueness of such solutions is an open problem.

We sall require an additional property of generalized solutions in the sequel. To this end, let us introduce the *Stokes problem*

$$\begin{cases} \dfrac{dz(t)}{dt} + Az(t) = \dfrac{d\omega(t)}{dt}, & t \in [0, T] \\ z(0) = 0. \end{cases} \tag{7}$$

Given $\omega \in \Omega_0$, this equation has a unique solution $z \in C([0, T]; D(A^{\alpha - \varepsilon}))$ for all $\varepsilon > 0$, where α is any number in $(\frac{3}{8}, \frac{1}{2})$ satisfying the Sobolev embedding $\Omega_0 \subset C^\alpha([0, T]; H)$; and the mapping $\omega \mapsto z(\cdot; \omega)$ is linear continuous from Ω_0 to $C([0, T]; D(A^{\alpha - \varepsilon}))$. See [13] for the details. The solution is explicitly given by

$$z(t, \omega) = e^{-tA}\omega(t) + \int_0^t Ae^{-(t-s)A}(\omega(t) - \omega(s))\, ds,$$

and it is a solution in the sense that

$$\langle z(t), \phi \rangle + \int_0^t \langle z(s), A\phi \rangle\, ds = \langle \omega(t), \phi \rangle \tag{8}$$

for all $\phi \in D(A)$ and all $t \geqslant 0$. The reader my easily reconstruct a sketch of the proof of these results by means of the following estimate:

$$|A^{\alpha - \varepsilon} z(t)| \leqslant \frac{C}{t^{\alpha - \varepsilon}} |\omega(t) - \omega(0)| + \int_0^t \frac{C}{(t-s)^{1 + \alpha - \varepsilon}} |\omega(t) - \omega(s)|\, ds,$$

where the α-Hölder continuity of ω has to be used. The estimate on the semigroup used above can be found in [28].

In particular we have $z \in C([0, T]; D(A^{3/8}))$. This implies that $z \in C([0, T]; [L^4(D)]^3)$, which we shall often use.

Using the process z, we shall impose on generalized solutions the following additional hypothesis:

(H) (*reducing if necessary the set Ω_{00} of the previous definition*) *for all $\omega \in \Omega_{00}$ the function $v(\cdot, \omega)$ defined as $v(t, \omega) = u(t, \omega) - z(t, \omega)$ belongs to*

$L^2(0, T; V)$ *(beside the property* $v(\cdot, \omega) \in L^\infty(0, T; H) \cap C([0, T]; D(A^{-1}))$ *which holds true by definition), and it satisfies the energy-type inequality*

$$\frac{1}{2}|v(t)|^2 + \int_0^t \|v(s)\|^2\, ds$$

$$\leqslant \frac{1}{2}|u_0|^2 + \int_0^t \langle B(u(s), v(s)), u(s)\rangle\, ds + \int_0^t \langle f, v(s)\rangle\, ds \tag{9}$$

for all $t \in [0, T]$.

Often, in similar definitions, the energy inequality is assumed to hold for almost every $t \in (0, T)$ only. However, since $v \in C([0, T]; H_w)$ (see the remark following Definition 1.1), when inequality (9) holds for almost every $t \in (0, T)$ then it holds for all $t \in [0, T]$.

The reader can easily check that (9) follows from the definition of v and the equations for u and z, by differentiation of $|v(t)|^2$; however, this computation is only formal, so the energy inequality has to be imposed. This fact is similar to the deterministic case.

The validity of property (H) seems to hold in all the cases treated by the authors mentioned above, but is not explicitly stated in the literature, except for [17]. In [17] it is proved in detail that there exists a generalized solution which satisfies (H).

We conclude this subsection by deducing two estimates from the energy inequality in (H). We have, using (3), (4), and the identity $u = v + z$,

$$\frac{1}{2}|v(t)|^2 + \int_0^t \|v(s)\|^2\, ds$$

$$\leqslant \frac{1}{2}|u_0|^2 + C_B \int_0^t \|v\|\, |v+z|_{L^4(D)}\, |z|_{L^4(D)}\, ds + \int_0^t |f|\, |v(s)|\, ds$$

$$\leqslant \frac{1}{2}|u_0|^2 + \frac{1}{4}\int_0^t \|v\|^2\, ds + C_1 \int_0^t (|v|^2_{L^4(D)} + |z|^2_{L^4(D)})\, |z|^2_{L^4(D)}\, ds$$

$$+ t\,|f|^2 + \int_0^t |v(s)|^2\, ds$$

$$\leqslant \frac{1}{2}|u_0|^2 + \frac{1}{4}\int_0^t \|v\|^2\, ds + C_1 \int_0^t C_S \|v\|^{3/2}\, |v|^{1/2}\, |z|^2_{L^4(D)}\, ds$$

$$+ C_1 \int_0^t |z|^4_{L^4(D)}\, ds + t\,|f|^2 + \int_0^t |v(s)|^2\, ds$$

$$\leqslant \frac{1}{2}|u_0|^2 + \frac{1}{2}\int_0^t \|v\|^2\, ds + C_2 \int_0^t |v|^2\, |z|^8_{L^4(D)}\, ds$$

$$+ C_1 \int_0^t |z|^4_{L^4(D)}\, ds + t\,|f|^2 + \int_0^t |v(s)|^2\, ds$$

for suitable positive constants C_1 and C_2. Therefore

$$|v(t)|^2 + \int_0^t \|v(s)\|^2\, ds$$
$$\leqslant |u_0|^2 + 2C_1 \int_0^T |z|^4_{L^4(D)}\, ds + 2T\,|f|^2 + \int_0^t |v|^2\, 2(C_2\, |z|^8_{L^4(D)} + 1)\, ds. \tag{10}$$

By Gronwall lemma we get

$$|v(t)|^2 \leqslant \left(|u_0|^2 + 2C_1 \int_0^T |z|^4_{L^4(D)}\, ds + 2T\,|f|^2 \right) e^{\int_0^T 2(C_2\,|z|^8_{L^4(D)}+1)\, ds},$$
$$t \in [0, T]. \tag{11}$$

Inserting this estimate in (10) we also get

$$\int_0^T \|v\|^2\, ds \leqslant \left(|u_0|^2 + 2C_1 \int_0^T |z|^4_{L^4(D)}\, ds + 2T\,|f|^2 \right)$$
$$\times \left(1 + e^{\int_0^T 2(C_2\,|z|^8_{L^4(D)}+1)\, ds} \int_0^T 2(C_2\,|z|^8_{L^4(D)} + 1)\, ds \right), \qquad t \in [0, T]. \tag{12}$$

We shall use this inequality in the proof of Lemma 3.1.

1.3. *Main Result*

We have remarked above that for a generalized solution we have $u(\cdot, \omega) \in C([0, T]; H_w)$ for all $\omega \in \Omega_{00}$. It can be proved by the same method that, given $t \in [0, T]$, the well defined mapping $\omega \mapsto u(t, \omega)$ from Ω_{00} to H is also measurable. Indeed, following the proof of [30], $u(t, \omega)$ is the weak limit in H of $(\eta_\varepsilon \circ u(\cdot, \omega))(t)$, where η_ε are classical mollifiers; and the mapping $\omega \mapsto (\eta_\varepsilon \circ u(\cdot, \omega))(t)$ is measurable from Ω_{00} to H since the mapping $\omega \mapsto u(\cdot, \omega)$ is measurable from Ω_{00} to $L^2(0, T; H)$.

Thus, given a generalized solution $u(\cdot, \omega; u_0)$ with initial condition u_0, it is well defined the image measure $P(t, u_0, \cdot)$ of the mapping $\omega \mapsto u(t, \omega)$ from Ω_{00} to H. In other words, we can define the *transition probabilities*

$$P(t, u_0, \Gamma) := P(u(t, \omega; u_0) \in \Gamma)$$

for all Borel sets Γ of H and all $t \in [0, T]$. Recall that the solution $u(t, \omega; u_0)$ may not be unique. Thus, $P(t, u_0, \Gamma)$ is not uniquely defined by the Eq. (5), but it depends on the solution $u(t, \omega; u_0)$ that we take; we do not use an explicit notation for this dependence, but we shall try to recall this fact in the main statements.

For the same reason, we do not know if the transition probabilities satisfy the Chapman–Kolmogorov equation. It seems that one can extract a Markov selection, but a complete proof has not been written in the literature. Anyway, we introduce the notion of irreducibility.

We say that a family of transition probabilities $\{P(t, u_0, \cdot);\ t \in [0, T],\ u_0 \in H\}$, constructed as above by means of generalized solutions of Eq. (5), is irreducible in H if for every $u_0 \in H$, $t \in (0, T]$, $x \in H$ and $\varepsilon > 0$, we have $P(t, u_0, B_H(x, \varepsilon)) > 0$. At present we cannot prove this form of irreducibility. Thus we introduce a weaker notion. We say that a family of transition probabilities $\{P(t, u_0, \cdot); t \in [0, T], u_0 \in H\}$, constructed as above with generalized solutions of Eq. (5), is (V, H)-*irreducible* if for every $u_0 \in V$, $t \in (0, T]$, $x \in H$ and $\varepsilon > 0$, we have $P(t, u_0, B_H(x, \varepsilon)) > 0$.

Remark. Using a notion of generalized solution defined for all $t \geqslant 0$, one can give a definition of transition probabilities and irreducibility for $t \geqslant 0$ and not only over a finite time interval $[0, T]$. However, the proof of existence of generalized solutions defined for all $t \geqslant 0$ is not common in the literature (it is given in [17]). Since there is no difference for our purposes, we restrict ourselves to the more common framework of solutions on $[0, T]$.

Given $\omega \in \Omega_0$ and $\varepsilon > 0$, we denote by $B_{W^{s,p}([0,T];H)}(\omega, \varepsilon)$ the ball in $W^{s,p}([0, T]; H)$ of center ω and radius ε.

The aim of this paper is to prove the following result.

THEOREM 1.1. *Assume that the measure P is full in Ω_0, in the sense that for every $\omega \in \Omega_0$ and every $\varepsilon > 0$ one has $P(B_{W^{s,p}([0,T];H)}(\omega, \varepsilon)) > 0$.*

Let $\{P(t, u_0, \cdot); t \in [0, T], u_0 \in H\}$ be a family of transition probabilities, constructed as above by means of some arbitrary generalized solutions of Eq. (5), *and assume that these generalized solutions satisfy condition* (*H*).

Then the family $\{P(t, u_0, \cdot); t \in [0, T], u_0 \in H\}$ is (V, H)-irreducible.

Remark. We have used the language of irreducibility since this property plays a role in ergodic theory. But we could also restate the conclusion of Theorem 1.1 simply as follows: for every $u_0 \in V$, and every generalized solution $u(\cdot, \omega; u_0)$ of Eq. (5) with initial condition u_0, satisfying property (H), the support in H of the law of $u(t, \cdot; u_0)$ is H (for $t \in (0, T]$).

As a particular case of P we can take a Wiener measure, as in Example 1.1. If $\sigma_n \neq 0$ for each n, then P is full in Ω_0 in the sense of the previous statement. Indeed, the reproducing kernel space of P as a Gaussian measure on Ω_0 (being the same as the reproducing kernel space of P as a Gaussian measure on $L^2(0, T; H)$) is the space

$$K := \left\{ v \in H^1(0, T; H) \mid v(0) = 0, \frac{dv(t)}{dt} = Q^{1/2} u(t), \text{ for some } u \in L^2(0, T; H) \right\}.$$

To show that P is full in Ω_0 we have to prove that K is dense in Ω_0, endowed with the topology of $W^{s,p}(0, T; H)$. The proof goes as follows: using classical mollifiers one checks that $K_1 := \{v \in H^1(0, T; H) \mid v(0) = 0\}$ is dense in Ω_0; then, using finite dimensional projections in H over the span of a finite number of functions ϕ_n, one shows that K is dense in K_1 endowed with the $H^1(0, T; H)$ topology, which is stronger than the topology of $W^{s,p}(0, T; H)$; this completes the proof. Then we have:

Corollary 1.1. *Let P be a Wiener measure, as in Example* 1.1. *Assume that the incremental covariance Q in H of the Wiener measure P is injective. In other words, assume that $\sigma_n \neq 0$ for each n.*

Let $\{P(t, u_0, \cdot); t \in [0, T], u_0 \in H\}$ be a family of transition probabilities, constructed as above by means of some arbitrary generalized solutions of Eq. (5), *and assume that these generalized solutions satisfy condition* (H).

Then the family $\{P(t, u_0, \cdot); t \in [0, T], u_0 \in H\}$ is (V, H)-irreducible.

Theorem 1.1 will be proved in Section 4 after two preparing sections. Let us give a comment on the strategy of proof.

In the simplest cases, in order to prove the irreducibility, one has to prove an approximate controllability property for the mapping $\omega \mapsto u(t, \omega)$, for every given t, and the continuity of this mapping. The controllability property will be proved in Section 2, and it is not surprising. What is less trival, in principle, is the continuity of the mapping. Indeed, a major open problem in the theory of 3-D Navier–Stokes equation is the well posedness of the equation (uniqueness of weak solutions and continuous dependence on data are open problems). But here we need to prove the continuous dependence on ω only for those particular solutions constructed in Section 2 in the controllability argument. *These solutions are regular*. It is well known that regular solutions, when they exist, are unique also in the class of weak solutions (see [24] for a proof of this classical result). Here we prove a similar property: the continuous dependence on ω around regular solutions.

This property has no counterpart for finite dimensional systems, for instance. Thus, in some sense, the result of irreducibility proved here is typical of equations like the Navier–Stokes equation, that are *well posed around regular solutions*, while it is not a general property of non-well-posed systems with full-measure noise.

2. APPROXIMATE CONTROLLABILITY

The following result is a property of approximate controllability in the space H. Notice however that we assume $u_0 \in V$.

LEMMA 2.1. *Let $T>0$, $f\in H$, $u_0\in V$, and $u_T\in D(A)$ be given. Then there exist*

$$\bar{\omega}\in Lip([0,T];H),\ \bar{u}\in C([0,T];V)\cap L^2(0,T;D(A))$$

such that

(a) *$\bar{u}$ is a solution of the deterministic Eq. (5) with $\omega=\bar{\omega}$, in the sense that for all $t\in[0,T]$ we have*

$$\bar{u}(t)+\int_0^t A\bar{u}(s)\,ds+\int_0^t B(\bar{u}(s),\bar{u}(s))\,ds=u_0+\int_0^t f\,ds+\bar{\omega}(t) \tag{13}$$

the identity being in H,

(b) $\bar{u}(T)=u_T$,

(c) *if $\bar{z}$ denote the solution to (7) corresponding to $\bar{\omega}$, and $\bar{v}$ is defined as $\bar{v}=\bar{u}-\bar{z}$, then*

$$\bar{v}\in C([0,T];V)\cap L^2(0,T;D(A)).$$

Proof. *Step* 1. Let us consider Eq. (5) with $\omega=0$. Since $u(0)\in V$, there exists $T^*\in(0,T)$ such that Eq. (5) has a unique solution $u\in C([0,T^*];V)\cap L^2(0,T^*;D(A))$ (cf. [34, 35] or other classical references on 3-D Navier–Stokes equation). Thus $u(t)\in D(A)$ for a.e. $t\in[0,T^*]$. Therefore, passing to a smaller value of T^* if necessary, we can always assume that $u(T^*)\in D(A)$. We denote by $\bar{u}(t)$ the solution just found for $t\in[0,T^*]$, and by $\bar{\omega}(t)$ the function identically equal to zero defined for $t\in[0,T^*]$.

Step 2. Let us now work on $[T^*,T]$ (we have chosen $T^*<T$ above). We define $\bar{u}(t)$ for $t\in[T^*,T]$ as the linear interpolation of $\bar{u}(T^*)$ (found in Step 1) and u_T:

$$\bar{u}(t)=\frac{T-t}{T-T^*}\bar{u}(T^*)+\frac{t-T^*}{T-T^*}u_T,\qquad t\in[T^*,T].$$

We have $\bar{u}\in C([T^*,T];D(A))$. Hence

$$\xi:=\frac{d\bar{u}}{dt}+A\bar{u}+B(\bar{u},\bar{u})-f\in D(A)+C([T^*,T];H)$$

$$+L^\infty(T^*,T;H)+H\subset L^\infty(T^*,T;H). \tag{14}$$

We have only to clarify why $B(\bar{u}, \bar{u}) \in L^\infty(T^*, T; H)$. This holds true since for every $\phi \in V$ we have

$$
\begin{aligned}
|\langle B(\bar{u}(t), \bar{u}(t)), \phi \rangle| &\leqslant C_1 \, |\phi| \, \|\bar{u}(t)\| \, |\bar{u}(t)|_{L^\infty(D)} \\
&\leqslant C_2 \, |\phi| \, \|\bar{u}(t)\| \, |A\bar{u}(t)| \qquad (15)
\end{aligned}
$$

for some constants C_1, C_2, by the Sobolev embedding theorem.

Define now

$$
\bar{\omega}(t) = \int_{T^*}^{t} \xi(s) \, ds; \qquad (16)
$$

we have $\bar{\omega} \in Lip([T^*, T]; H)$. Merging with the function $\bar{\omega}$ found in Step 1, we readily have $\bar{\omega} \in Lip([0, T]; H)$. Also the function $\bar{u}$ found by connecting the two corresponding functions of Steps 1 and 2 satisfy the regularity required by the Lemma, and point (b). As to point (a), it holds true for $t \in [0, T^*]$ by definition of $\bar{u}$ and $\bar{\omega}$, and for $t \in [T^*, T]$ by (14) and (16).

To prove (c) notice that

$$
\frac{d\bar{v}}{dt} + A\bar{v} = -B(\bar{u}, \bar{u}) + f \in L^2(0, T; H)
$$

(the property $B(\bar{u}, \bar{u}) \in L^2(0, T; H)$ follows from (15) and the regularity of $\bar{u}$) and $\bar{v}(0) = u_0 \in V$. Hence $\bar{v} \in C([0, T]; V) \cap L^2(0, T; D(A))$ by a classical regularity result (cf. [25]). The proof is complete.

Remark. With the same line of proof of Lemma 3.1 below, one can show that a solution u with the same reglarity as $\bar{u}$ of the previous lemma, is unique, also in the class of weak solutions.

3. CONTINUITY OF THE SOLUTION MAPPING ALONG THE CONTROLLERS

In the previous section we proved the approximate controllability result using only regular solutions. Here we prove the continuity of the mapping $\omega \mapsto u(T, \omega)$ around the functions that are involved in the controllability procedure, or, more generally, around regular solutions. This result is conceptually similar to the well known fact that regular solutions, when they exist, are unique in the class of weak solutions.

LEMMA 3.1. *Let* $f \in H$, $u_0 \in V$, $t_f \in [0, T]$, *and* $\omega \in \Omega_0$ *be given. Let*

$$u \in C([0, t_f]; V) \cap L^2(0, t_f; D(A))$$

be a solution on $[0, t_f]$ *of Eq.* (13) *corresponding to* ω, *such that* $v := u - z \in C([0, t_f]; V) \cap L^2(0, t_f; D(A))$, *where* z *is the solution of* (7) *corresponding to* ω.

Let $u(\cdot, \omega)$, $\omega \in \in \Omega_{00}$ *be a generalized solution of Eq.* (5) (*defined at least on* $[0, t_f]$) *satisfying the assumption* (*H*).

Let $\omega_n \in \Omega_{00}$ *be a sequence converging to* ω *in* $W^{s,p}([0, t_f]; H)$. *Then* $u(t, \omega_n)$ *converges to* $u(t)$ *in* H, *uniformly in* $t \in [0, t_f]$.

Proof. Let $u_n(t) = u(t, \omega_n)$, $v(t) = u(t) - z(t)$, $v_n(t) = u_n(t) - z_n(t)$, where $z(t)$ and $z_n(t)$ are the solutions of Eq. (7) corresponding to ω and ω_n respectively. Since $u(\cdot, \omega)$ satisfies (H), for all $t \in [0, t_f]$

$$\frac{1}{2} |v_n(t)|^2 + \int_0^t \|v_n(s)\|^2 \, ds$$

$$\leqslant \frac{1}{2} |u_0|^2 + \int_0^t \langle B(u_n(s), v_n(s)), u_n(s) \rangle \, ds + \int_0^t \langle f, v_n(s) \rangle \, ds. \quad (17)$$

Moreover, from Eq. (13) and the regularity of u and v we obtain for all $t \in [0, t_f]$

$$\frac{1}{2} |v(t)|^2 + \int_0^t \|v(s)\|^2 \, ds$$

$$= \frac{1}{2} |u_0|^2 + \int_0^t \langle B(u(s), v(s)), u(s) \rangle \, ds + \int_0^t \langle f, v(s) \rangle \, ds. \quad (18)$$

Finally, using again the regularity of u and v, and the weak formulation (6) of Definition 1.1, we get for all $t \in [0, t_f]$

$$\langle v(t), v_n(t) \rangle = |u_0|^2 + \int_0^t \left\langle \frac{dv}{ds}, v_n(s) \right\rangle ds - \int_0^t \langle Av(s), v_n(s) \rangle \, ds$$

$$+ \int_0^t \langle B(u_n(s), v(s)), u_n(s) \rangle \, ds + \int_0^t \langle f, v(s) \rangle \, ds. \quad (19)$$

We replace dv/ds by $-Av - B(u, u) + f$ in (19), add (17) and (18) and subtract (19). The result is that for all $t \in [0, t_f]$

$$\tfrac{1}{2}\,|v(t)-v_n(t)|^2+\int_0^t \|v(s)-v_n(s)\|^2\,ds$$

$$\leqslant \int_0^t (\langle B(u_n, v_n), u_n\rangle + \langle B(u, v), u\rangle - \langle B(u_n, v), u_n\rangle + \langle B(u, u), v_n\rangle)\,ds$$

$$= \int_0^t (\langle B(u_n, v_n - v), u_n\rangle - \langle B(u, v_n - v), u\rangle)\,ds, \tag{20}$$

where we have also used the second identity in (2). The last integrand is formally equal to $-\langle B(u, u) - B(u_n, u_n), v - v_n\rangle$. By simple manipulations, the first property in (2), and (3), we have

$$\langle B(u_n, v_n - v), u_n\rangle - \langle B(u, v_n - v), u\rangle = \langle B(u_n, v_n - v), u_n - u\rangle$$

$$+ \langle B(u_n - u, v_n - v), u\rangle$$

$$= \langle B(u_n, v_n - v), z_n - z\rangle + \langle B(u_n - u, v_n - v), u\rangle$$

$$\leqslant C_B\,\|v - v_n\|\,|u_n|_{L^4(D)}\,|z - z_n|_{L^4(D)} + C_B\,\|v - v_n\|\,|v - v_n|_{L^4(D)}\,|u|_{L^4(D)}$$

$$+ C_B\,\|v - v_n\|\,|z - z_n|_{L^4(D)}\,|u|_{L^4(D)}$$

and using also the regularity of u, which implies, by the Sobolev embedding theorem, that $\sup_{0\leqslant t\leqslant t_f} |u(t)|_{L^4(D)} < \infty$,

$$\leqslant \tfrac{1}{4}\,\|v - v_n\|^2 + C_1\,|v_n + z_n|^2_{L^4(D)}\,|z - z_n|^2_{L^4(D)}$$

$$+ C_1\,|v - v_n|^2_{L^4(D)} + C_1\,|z - z_n|^2_{L^4(D)}$$

for some positive constant C_1. Therefore

$$\tfrac{1}{2}\,|v(t)-v_n(t)|^2+\tfrac{3}{4}\int_0^t \|v(s)-v_n(s)\|^2\,ds$$

$$\leqslant C_1\beta_n \int_0^t |v_n + z_n|^2_{L^4(D)}\,ds + C_1 \int_0^t |v - v_n|^2_{L^4(D)}\,ds + C_1\,T\beta_n,$$

where

$$\beta_n := \sup_{0\leqslant t\leqslant t_f} |z(t) - z_n(t)|^2_{L^4(D)}.$$

By (4) we finally have

$$\tfrac{1}{2}\,|v(t)-v_n(t)|^2+\tfrac{3}{4}\int_0^t \|v(s)-v_n(s)\|^2\,ds$$

$$\leqslant C_1\beta_n\int_0^t |v_n+z_n|^2_{L^4(D)}\,ds$$

$$+C_1\int_0^t C_S^2\,|v-v_n|^{1/2}\,\|v-v_n\|^{3/2}\,ds+C_1T\beta_n$$

$$\leqslant C_1\beta_n\int_0^{t_f} |v_n+z_n|^2_{L^4(D)}\,ds$$

$$+\tfrac{1}{4}\int_0^t \|v-v_n\|^2\,ds+C_2\int_0^t |v-v_n|^2\,ds+C_1T\beta_n$$

for some positive constant C_2. Thus

$$|v(t)-v_n(t)|^2+\int_0^t \|v(s)-v_n(s)\|^2\,ds$$

$$\leqslant 2C_1\beta_n\int_0^{t_f} |v_n+z_n|^2_{L^4(D)}\,ds+2C_1T\beta_n+2C_2\int_0^t |v-v_n|^2\,ds.$$

By the Gronwall lemma (we can neglect the term $\int_0^t \|v(s)-v_n(s)\|^2\,ds$)

$$|v(t)-v_n(t)|^2\leqslant\left(2C_1\beta_n\int_0^{t_f} |v_n+z_n|^2_{L^4(D)}\,ds+2C_1T\beta_n\right)e^{2C_2t}. \tag{21}$$

First recall that the mapping $\omega\mapsto z$ defined by Eq. (7) is continuous from $W^{s,p}([0,t_f];H)$ to $C([0,t_f];D(A^{3/8}))$, and $D(A^{3/8})\subset[L^4(D)]^3$. Therefore, since $\omega_n\to\omega$ in $W^{s,p}([0,t_f];H)$, we have

$$\beta_n\to 0,\ \sup_{n\geqslant 0}\int_0^{t_f} |z_n|^2_{L^4(D)}\,ds<\infty.$$

Second, since $V\subset[L^4(D)]^3$, by $\sup_{n\geqslant 0}\sup_{0\leqslant t\leqslant t_f}|z_n(t)|_{L^4(D)}<\infty$ and inequality (12) we have

$$\sup_{n\geqslant 0}\int_0^{t_f} |v_n|^2_{L^4(D)}\,ds<\infty.$$

Collecting these facts, (21) implies that $v_n(t)$ converges to $v(t)$ in H, uniformly in $t \in [0, t_f]$. Since $z_n \to z$ in $C([0, t_f]; D(A^{3/8}))$, we finally have that $u_n(t)$ converges to $u(t)$ in H, uniformly in $t \in [0, t_f]$. The proof is complete.

4. IRREDUCIBILITY (PROOF OF THEOREM 1.1)

We can now prove Theorem 1.1. The proof is classical, using the results of the previous sections. Let $u_0 \in V$, $t_f \in (0, T]$, $x \in H$ and $\varepsilon > 0$, be given, and let $u(t; \omega)$, $t \in [0, t_f]$, $\omega \in \Omega_{00}$, be a generalized solution of Eq. (5) satisfying (H), with initial condition u_0.

Let $u_{t_f} \in D(A)$ be a point such that $|x - u_{t_f}| < \varepsilon/2$. Then

$$P(t_f, u_0, B_H(x, \varepsilon)) \geqslant P(t_f, u_0, B_H(u_{t_f}, \varepsilon/2)).$$

By Lemma 2.1 (with $T = t_f$) there exists $\bar{\omega} \in Lip([0, t_f]; H)$ and $\bar{u} \in C([0, t_f]; V) \cap L^2(0, t_f; D(A))$ such that $\bar{u}$ is a solution of Eq. (5) over $[0, t_f]$ corresponding to $\bar{\omega}$, with initial condition u_0, and with $\bar{u}(t_f) = u_{t_f}$. By Lemma 3.1, there exists $\delta > 0$ such that for all $\omega \in \Omega_{00}$ with

$$\|\omega - \bar{\omega}\|_{W^{s,p}([0, t_f]; H)} < \delta$$

we have

$$|u(t_f, \omega) - u_{t_f}| = |u(t_f, \omega) - \bar{u}(t_f)| < \varepsilon.$$

Therefore

$$P(t_f, u_0, B_H(u_{t_f}, \varepsilon/2)) \geqslant P(\|\omega - \bar{\omega}\|_{W^{s,p}([0, t_f]; H)} < \delta).$$

But P is full, so the latter probability is strictly positive. This proves the claim of Theorem 1.1.

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